CS 4414 Midterm Exam   Due 03/15/2012

This Exam is being given under the guidelines of the **Honor Code**. You are expected to respect those guidelines and to report those who do not. There are 4 questions for a total of 100 points.

1. Define $f(x) = \int_0^x \frac{\log(1+t)}{t}\,\mathrm{d}t$.
   (a) Obtain a polynomial approximation for the integrand about $t = 0$. Using this, derive a polynomial approximation for $f(x)$ about $x = 0$.
   (b) Bound the error in approximation to $f(x)$, with $|x| \le 1$, as a function of the degree n of the polynomial.
   (c) Compute the integral for several values of n, at different values of the argument x and assess the accuracy of the approximation, given that

$$f(0.01) = 0.00997511, \quad f(0.1) = 0.0976052, \quad f(0.5) = 0.448414, \quad f(1) = \frac{\pi^2}{12}\,.$$

2. (a) In some situations, loss-of-significance errors can be avoided by rearranging the function being evaluated. Show how to avoid such errors in the following function, where x is near 0:

$$f(x) = \frac{1-\cos(x)}{x^2}\,.$$

   (b) Write a matlab program to compute the function value for argument values $x = 1.e-3 \cdot i$, for all $i = 1, \cdots, 100$. Using Matlab's *semilogy* function plot the error (the difference between the two formulas) as a function of x. Discuss the results.

3. Calculate the condition number of the following matrix

$$A = \begin{bmatrix} 1 & c \\ c & 1 \end{bmatrix}$$

   using $\|\cdot\|_\infty$.

   How well- or ill-conditioned if the matrix for $0 < c < 1$?

   Explain this conditioning *geometrically*.

   Hint: $\|A\|_\infty = max_i \sum_j |a_{i,j}|$.

4. (a) Consider the *Hilbert* matrix, whose entries are

$$H_{i,j} = \frac{1}{i+j-1}\,, \quad 1 \le i,j \le n\,.$$

   Consider also the vector

$$b_i = \sum_{j=1}^{n} \frac{1}{i+j-1}\,, \quad 1 \le i \le n\,.$$

   Write a program that solves the system

$$H\,x = b$$

   You can use Matlab.

   Bonus points: write the program in C or Fortran using calls to LAPACK library (the functions *dgetrf* and *dgetrs*).

(b) What is the exact solution x_{exact} of the system?

Solve the system repeatedly for different values of n, $2 \le n \le 12$. How accurate are the numerical solutions x_{numeric}? Tabulate the solution errors

$$e(n) = \frac{\|x_{\text{numeric}} - x_{\text{exact}}\|}{\|x_{\text{exact}}\|}$$

as a function of n. You can plot errors versus n using matlab's *semilogy* function. Explain in detail the results.