CS–4414, Spring 2012
Final exam

This Exam is being given under the guidelines of the **Honor Code**. You are expected to respect those guidelines and to report those who do not. Good luck!

1. Consider the integral $\int_a^b f(x)\,dx$ for a smooth function $f(x)$. Suppose you compute the above integral with points $x_0, x_1, \ldots, x_n$. Let the error be denoted by E_n and next you compute with points $x_0, x_{\frac{1}{2}}, x_1, \ldots, x_{n-1}, x_{n-\frac{1}{2}}, x_n$ and let the error be denoted by E_{2n}. The nodes are equispaced. What is the ratio of errors E_n/E_{2n} in case of
   (a) Simpsons method
   (b) Trapezoidal rule
   (c) Explain the concept of condition number of a matrix A. Explain what it tells about the error in solving $Ax = b$ when there are small errors ΔA, Δb in the coefficients

2. Consider the integral
$$\int_{-1}^{1} \frac{1}{1+x^2}\,dx = \frac{\pi}{2}\,.$$
   Compute this integral numerically using various numbers of points with the following methods:
   (a) Trapezoidal rule
   (b) Simpsons method
   (c) Gaussian quadrature

   How many points do you need for each method to obtain 4 accurate digits?

   Plot the logarithm of the approximation error versus the number of points for each of the methods. What do you conclude? Discuss in detail.

3. Use the method of undetermined coefficients in order to approximate:
   (a) $f''(x_0)$ given $f(x_0), f(x_1), f(x_2)$, where $x_1 = x_0 + h$ and $x_2 = x_1 + h$
   (b) $f''(x_0)$ given $f(x_0), f(x_1), f(x_2), f(x_3)$, where $x_1 = x_0 + h$, $x_2 = x_0 - h$, $x_1 = x_0 + h$, $x_3 = x_0 + 2h$

4. Choose your own a linear system $Ax = b$ of dimension $n \geq 10$. Let D be the diagonal part of A, L the strict lower triangle of A (excluding the diagonal), and U the strict upper triangle of A (excluding the diagonal).
   (a) Perform 10 steps of Jacobi iteration (starting from $x^{(0)} = 0$)
   (b) Perform 10 steps of Gauss-Seidel iterations (starting from $x^{(0)} = 0$)
   (c) Perform 10 steps of successive over relaxation (starting from $x^{(0)} = 0$). This method uses iterations of the form:
$$(D + \omega\,L)\,x^{(k+1)} = \omega\,b - \big((\omega-1)\,D + \omega\,U\big)\,x^{(k)}$$
   with $\omega = 1.8$.

(d) Compare the solutions obtained in each case with the actual solution obtained by LU decomposition. Plot the logarithm of the solution error norm versus the iteration number for each of the methods. Discuss.

5. Solve the integral equation

$$\int_0^1 \sqrt{s^2+t^2} \cdot y(t) \cdot dt = \frac{(s^2+1)^{3/2} - s^3}{3}$$

for the unknown function $y(t)$. The exact result is $y(t) = t$.

(a) Approximate the left hand side integral by Simpson formula with equally spaced points $t_0, \ldots, t_n$.

(b) You obtain a linear system in the unknown variables $y_i \approx y(t_i)$, $i = 0, \ldots, n$.

(c) Solve this system. Plot the error of the solution (against the exact result) for different values of n. Comment on the results.

6. **(Extra credit)** Consider the gamma function

$$\Gamma(x) = \int_0^\infty t^{x-1}\, e^{-t}\, dt$$

which equals the factorial for integer arguments:

$$\Gamma(n) = n!$$

(a) Consider the points $(n, n!)$ for n=1,2,3,4,5,6. Compute the cubic spline interpolant (you can use matlab's built in `spline` function).

(b) Plot the gamma function (`gamma`, in matlab) versus the spline interpolant.

(c) Plot the error and comment.

7. **(Extra credit)** Approximate integrals of the form

$$\int_{-1}^{1} f(x)\, x^2\, dx = w_0 f(x_0) + w_1 f(x_1)$$

for any function $f(x)$. Find x_0, x_1, w_0, w_1 such that the above formula integrates exactly polynomials of degree up to 3.

Hint: Note that the integral has an extra term x^2. The Gaussian quadrature technique has to be generalized for this case.