How to Solve Hard Problems by Making Them Harder

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Research Areas

- Numerical analysis
- Nonlinear programming
- Parallel computing
- Mathematical software
- Image processing/computer vision
- Solid mechanics
- Fluid mechanics
- Computational biology/bioinformatics
- Multidisciplinary engineering design
- Scientific computing

Center Memberships

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SEMI     CSSS     
Nonlinear programming is

1) unstructured computer programming using GO TOs;

2) a variant of extreme programming;

3) how computers were programmed in the 1960s;

4) the study of nonlinear constrained optimization problems.
Homotopy Algorithms

Let $x = (x_1, \ldots, x_n) \in E^n$, $F : E^n \to E^n$ be $C^2$.

Hard problem: $F(x) = 0$.

Newton’s method: $x^{(k+1)} = x^{(k)} - (DF(x^{(k)}))^{-1} F(x^{(k)})$.

Homotopy method: let $S(x) = 0$ be a simple problem with a unique solution $a$ (e.g., $S(x) = x - a = 0$).

Construct the homotopy map

$$\rho_a(\lambda, x) = \lambda F(x) + (1 - \lambda) S(x), \quad 0 \leq \lambda \leq 1.$$ 

Track the zero curve $\gamma$ of $\rho_a(\lambda, x) = 0$ from $\lambda = 0$ to $\lambda = 1$.

Under certain assumptions on $F$ and $S$, this works with probability one (for almost all $a$).
Linear Complementarity Problem (LCP)

\[ w - Mz = q, \quad w \geq 0, \quad z \geq 0, \quad w^t z = 0. \]

\[ Ax = q, \quad x \geq 0, \quad A_i \in \{I_i, -M_i\}, \quad 2^n \text{ systems.} \]

\[ K_i(z) = (Mz + q)_i^3 + z_i^3 - |(Mz + q)_i - z_i|^3 = 0 \]
\[ \iff w_i \geq 0, \quad z_i \geq 0, \quad w_i z_i = 0. \]

\[ \rho_a(\lambda, z) = \lambda K(z) + (1 - \lambda)(z - a), \quad 0 \leq \lambda \leq 1. \]
\[
\frac{d\theta}{ds} = Qx - Py + M, \quad \frac{dx}{ds} = \cos \theta, \quad \frac{dy}{ds} = \sin \theta.
\]

\[x(0) = y(0) = \theta(0) = 0,\]

\[x(1) = a, \quad y(1) = b, \quad \theta(1) = c.\]

\[v = \begin{pmatrix} Q \\ P \\ M \end{pmatrix}, \quad F(v) = \begin{pmatrix} x(1; v) - a \\ y(1; v) - b \\ \theta(1; v) - c \end{pmatrix} = 0.\]

\[\rho(\lambda, v) = \begin{pmatrix} x(1; v) - [\lambda a + (1 - \lambda)1] \\ y(1; v) - [\lambda b + (1 - \lambda)0] \\ \theta(1; v) - [\lambda c + (1 - \lambda)0] \end{pmatrix} = 0, \quad 0 \leq \lambda \leq 1.\]